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# THE ROTATING SHRINK FIT WITH ELASTIC–PLASTIC HUB EXHIBITING VARIABLE THICKNESS

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**Abstract**—Based on Tresca's yield condition, the associated flow rule and linear strain hardening, the stress distribution in the hub of a rotating shrink fit is studied. The rotating shrink fit consists of an elastic solid disk with uniform thickness and partially plasticized hub with variable thickness. It is assumed that the assembly is thin and that the variation of thickness is radial. The problem is solved in closed form.

## NOTATION

| | |
|---|---|
| $\sigma_{rr}$, $\sigma_{\theta\theta}$ | radial and circumferential stress components |
| $\sigma_0$, $\sigma_v$ | initial and subsequent yield stress respectively |
| $\varepsilon_{eq}$ | equivalent plastic strain |
| $d\varepsilon$ | strain increment |
| $u$ | radial displacement |
| $E$, $\nu$ | Young's modulus and Poisson's ratio |
| $\eta$ | work hardening parameter |
| $a$, $b$ | inner and outer radii of the hub |
| $q$ | $a/b$ radius ratio |
| $h$, $h_0$ | local thickness and thickness at $b$, respectively |
| $n$ | thickness parameter in $h = h_0(r/b)^{n}$ |
| $\omega$ | constant angular velocity of rotation |
| $\rho$ | mass density |
| $A$, $B$, $C$, $D$, $C_1$ | constants of integration |
| p | superscript denoting plastic component. |

It is convenient to introduce the following dimensionless quantities:

$$x = \frac{r}{b};\quad \xi = \frac{z}{b};\quad I = \frac{EI}{\sigma_0 a};\quad H = \frac{\eta\sigma_0}{E};\quad \bar{\sigma}_{ij} = \frac{\sigma_{ij}}{\sigma_0};\quad \bar{u} = \frac{Eu}{\sigma_0 b};\quad \Omega^2 = \frac{\rho\omega^2 b^2}{\sigma_0}.$$

## 1. INTRODUCTION

In the design of machinery (and some structures), the problem of transmitting a moment is frequently encountered. The significance of shrink fits to mechanical engineering lies in the fact that they are capable of transmitting high moments at low production costs. A simple computation reveals that by a purely elastic design the strength of especially the hub material is utilized poorly. This can be improved by an elastic–plastic design. The widespread use of shrink fits in machinery and structural applications has generated considerable interest in the application of the macroscopic theory of plasticity to engineering problems associated with shrink fits. The problem of rotating elastic–plastic shrink fits has been previously investigated by Kollmann (1981). His study is based on the Tresca's yield condition and the flow rule associated with it. A generalization of Kollmann's work for linear strain hardening materials has been given by Gamer (1986). Furthermore, it has been shown that hub material with an arbitrary nonlinear hardening law can be taken into account without much numerical calculation (Gamer, 1987a,b). Gamer and Kollmann (1986) developed a rigorous theory for a partly plasticized hub of perfectly plastic material. A hub material with perfectly plastic behavior up to a certain plastic strain and linear strain hardening for larger strain has been considered by Gamer (1987c). The case of a shrink fit with hollow inclusion and different materials has been solved by Müller (1989).

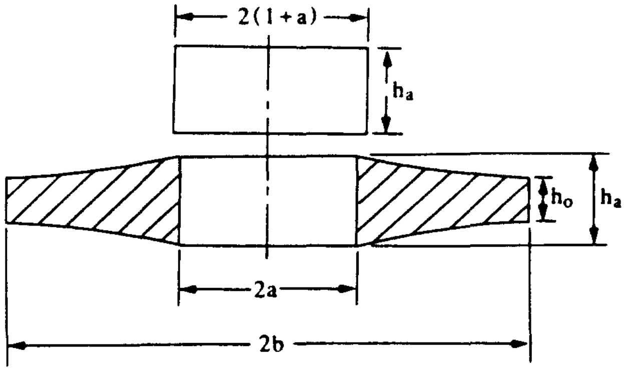


Fig. 1. Shrink fit geometry prior to assemblage.

In the above-mentioned papers consider the thickness of the hub as constant. The aim of this work is to develop the analytical solution for an elastic–plastic hub with variable thickness under the assumption of Tresca's yield condition, its associated flow rule and linear strain hardening. The inclusion is a circular solid disk with uniform thickness (Fig. 1). In addition, it is assumed that Young's modulus and Poisson's ratio, and the density of the solid disk and hub material, are equal.

## 2. BASIC EQUATIONS AND SOLUTION

We shall confine ourselves to the problem of purely elastic behavior of the solid disk and concentrate on the elastic–plastic behavior of the hub with variable thickness. Therefore, for the investigation under consideration, it is assumed that the radial stress does not become positive inside the plastic region due to rotation and the interferences.

For $a \leqslant r \leqslant z$ the hub material is plastic, while for $z \leqslant r \leqslant b$ it is still in an elastic state. We consider a state of plane stress and assume infinitesimal deformation. It is assumed that the variation of thickness is radial and is symmetric with reference to the midplane.

So long as there exist inequalities $\sigma_{rr} \leqslant 0$ and $\sigma_{\theta\theta} \geqslant 0$, the plastic deformation of the hub is governed by the yield condition

$$\sigma_{\theta\theta} - \sigma_{rr} = \sigma_y. \tag{1}$$

If the work-hardening law is taken to be

$$\sigma_y = \sigma_0(1 + \eta\varepsilon_{eq}) \tag{2}$$

where $\sigma_0$ is the initial tensile yield stress, $\eta$ is the hardening parameter and $\varepsilon_{eq}$ is an equivalent plastic strain. According to the flow rule associated with Tresca's yield condition

$$\mathrm{d}\varepsilon^{\mathrm{p}}_{rr} + \mathrm{d}\varepsilon^{\mathrm{p}}_{\theta\theta} = 0, \quad \mathrm{d}\varepsilon^{\mathrm{p}}_{zz} = 0. \tag{3}$$

Consideration of the equivalence of the increment of plastic work yields

$$\varepsilon_{eq} = \varepsilon^{\mathrm{p}}_{\theta\theta}. \tag{4}$$

For slowly varying angular velocity, the equation of motion

$$\frac{\mathrm{d}}{\mathrm{d}r}(hr\sigma_{rr}) - h\sigma_{\theta\theta} = -h\rho\omega^2 r^2 \tag{5}$$

and geometric relations

$$\varepsilon_{rr} = \frac{\mathrm{d}u}{\mathrm{d}r}, \quad \varepsilon_{\theta\theta} = \frac{u}{r} \tag{6}$$

hold in the entire hub irrespective of material behavior. Total strains are decomposed into elastic and plastic components. The stress–strain relations are:

$$\varepsilon_{rr} = \frac{1}{E}(\sigma_{rr} - \nu\sigma_{\theta\theta}) + \varepsilon_{rr}^{p} \tag{7}$$

$$\varepsilon_{\theta\theta} = \frac{1}{E}(\sigma_{\theta\theta} - \nu\sigma_{rr}) + \varepsilon_{\theta\theta}^{p}. \tag{8}$$

Since we restrict ourselves to small strains, $\varepsilon_{rr}$ and $\varepsilon_{\theta\theta}$ must satisfy the compatibility equation

$$\frac{\mathrm{d}}{\mathrm{d}r}(r\varepsilon_{\theta\theta}) = \varepsilon_{rr}. \tag{9}$$

Substituting the strains $\varepsilon_{rr}$ and $\varepsilon_{\theta\theta}$ in the compatibility equation (9) and using (1)–(5), we obtain

$$r^2\frac{\mathrm{d}^2\sigma_{rr}}{\mathrm{d}r^2} + \left(3 + r\frac{h'}{h}\right)r\frac{\mathrm{d}\sigma_{rr}}{\mathrm{d}r} + \left[\left(2 + \frac{1+\nu H}{1+H}\right)\frac{h'}{h} + r\left(\frac{h''}{h} - \frac{h'^2}{h^2}\right)\right]r\sigma_{rr} = \frac{2\sigma_0}{H+1} - \left(3 + \frac{1+\nu H}{H+1}\right)\rho\omega^2 r^2, \tag{10}$$

which is the differential equation expressed in terms of the radial stress, where $H = \eta\sigma_0/E$ and a prime denotes differentiation with respect to $r$.

If the thickness of the hub is assumed to vary along the radius in the form

$$h = h_0\left(\frac{r}{b}\right)^n, \tag{11}$$

and we substitute the thickness function (11) into eqn (10), the general solution of this equation is given by

$$\sigma_{rr} = Ar^{t_1} + Br^{t_2} - \frac{2\sigma_0}{nK_0} - \frac{K\rho\omega^2 r^2}{8(H+1) - nK}, \tag{12}$$

and using the equilibrium equation, the circumferential stress is found to be

$$\sigma_{\theta\theta} = t_{01}Ar^{t_1} + t_{02}Br^{t_2} - \frac{2(1-n)\sigma_0}{nK_0} - \frac{[4+(1+3\nu)H]}{8(H+1)-nK}\rho\omega^2 r^2 \tag{13}$$

where

$$t_{1,2} = \tfrac{1}{2}\left\{n - 2 \mp \sqrt{4 + n^2 + 4n(H\nu+1)/(H+1)}\right\}$$

$$t_{01} = t_1 + 1 - n, \quad t_{02} = t_2 + 1 - n, \quad K_0 = 2 + H(1+\nu), \quad K = 4 + H(3+\nu).$$

Combining the total circumferential strain–radial displacement relation and the total circumferential strain relation (8), and using (2) and (4), one obtains an expression for the radial displacement:

$$Eu = \left[\sigma_{\theta\theta} - \nu\sigma_{rr} + \frac{1}{H}(\sigma_y - \sigma_0)\right] r. \tag{14}$$

Inserting the stresses according to (12) and (13), and using (1), the preceding expression results in

$$Eu = \left\{\left[\frac{t_1 - n}{w^2} + 1 - \nu\right] A r^{t_1} + \left[\frac{t_2 - n}{w^2} + 1 - \nu\right] B r^{t_2} + \frac{\sigma_0(1-\nu)}{K_0}\left(1 - \frac{2}{n}\right) - \frac{(1-\nu)K_0}{8(H+1) - nk}\rho\omega^2 r^2\right\} r \tag{15}$$

where $w^2 = H/H+1$.

In the elastic region, $z \leqslant r \leqslant b$; the stresses and radial displacement are well known (see, for example, Timoshenko and Goodier, 1970) to be

$$\sigma_{rr} = C r^{\alpha+n-1} + D r^{\beta+n-1} - \frac{(3+\nu)\rho\omega^2 r^2}{8 - n(3+\nu)} \tag{16}$$

$$\sigma_{\theta\theta} = C\alpha r^{\alpha+n-1} + D\beta r^{\beta+n-1} - \frac{(1+3\nu)\rho\omega^2 r^2}{8 - n(3+\nu)} \tag{17}$$

$$Eu = \left\{C(\alpha - \nu) r^{\alpha+n-1} + D(\beta - \nu) r^{\beta+n-1} - \frac{(1-\nu^2)\rho\omega^2 r^2}{8 - n(3+\nu)}\right\} r \tag{18}$$

where

$$\alpha, \beta = \tfrac{1}{2}\left[-n \pm \sqrt{n^2 + 4(1 + n\nu)}\right].$$

In the elastic inclusion, $0 \leqslant r \leqslant a$; the stresses and radial displacement are given by

$$\sigma_{rr} = C_1 - \frac{(3+\nu)}{8}\rho\omega^2 r^2 \tag{19}$$

$$\sigma_{\theta\theta} = C_1 - \frac{(1+3\nu)}{8}\rho\omega^2 r^2 \tag{20}$$

$$Eu = \left[(1-\nu)C_1 - \frac{(1-\nu^2)}{8}\rho\omega^2 r^2\right] r. \tag{21}$$

## 3. THE ELASTIC-PLASTIC HUB

The above general expressions for stresses and displacements contain the unknowns constants $A$, $B$, $C$, $D$ and $C_1$. An additional unknown is the radius $z$ of the elastic–plastic interface. For the determination of these six unknowns there are six conditions available. The most convenient ones are: continuity of radial stress and displacement at $r = z$, continuity of radial stress $r = a$; the radial stress at the outer surface $r = b$ vanishes and at $r = a$ $u^{hub} - u^{disk} = I$, and, finally the yield condition in eqn (1) must be satisfied at $r = z$.

$$\bar{A} = \frac{A}{\sigma_0 b^{-t_1}} = \frac{1}{\left(\frac{t_1-n}{w^2}\right)\left[\left(\frac{q}{\xi}\right)^{t_1}-\left(\frac{q}{\xi}\right)^{t_2}\right]\xi^{t_1}}\left\{\bar{I}-\frac{1}{K_0}\left[1-\nu+(1+\nu)(H+1)\left(\frac{q}{\xi}\right)^{t_2}\right]\right.$$
$$\left.+\left[2(H+1)\left(\frac{q}{\xi}\right)^{t_2}\xi^2-\tfrac{1}{4}nKq^2\right]\frac{(1-\nu)\Omega^2}{8(H+1)-nK}\right\} \quad (22)$$

$$\bar{B} = \frac{B}{\sigma_0 b^{-t_2}} = \frac{1}{\left(\frac{t_2-n}{w^2}\right)\left[\left(\frac{q}{\xi}\right)^{t_2}-\left(\frac{q}{\xi}\right)^{t_1}\right]\xi^{t_2}}\left\{\bar{I}-\frac{1}{K_0}\left[1-\nu+(1+\nu)(H+1)\left(\frac{q}{\xi}\right)^{t_1}\right]\right.$$
$$\left.+\left[2(H+1)\left(\frac{q}{\xi}\right)^{t_1}\xi^2-\tfrac{1}{4}nKq^2\right]\frac{(1-\nu)\Omega^2}{8(H+1)-nK}\right\} \quad (23)$$

$$\bar{C} = \frac{C}{\sigma_0 b^{1-\alpha-n}} = \frac{1}{(\alpha-1)\xi^{\alpha+n-1}-(\beta-1)\xi^{\beta+n-1}}\left\{1-[(\beta-1)(3+\nu)\xi^{\beta+n-1}\right.$$
$$\left.+2(1-\nu)\xi^2]\frac{\Omega^2}{8-n(3+\nu)}\right\} \quad (24)$$

$$\bar{D} = \frac{D}{\sigma_0 b^{1-\beta-n}} = \frac{1}{[(\beta-1)\xi^{\beta+n-1}-(\alpha-1)\xi^{\alpha+n-1}]}\left\{1-[(\alpha-1)(3+\nu)\xi^{\alpha+n-1}\right.$$
$$\left.+2(1-\nu)\xi^2]\frac{\Omega^2}{8-n(3+\nu)}\right\} \quad (25)$$

$$\bar{C}_1 = \frac{C_1}{\sigma_0} = \bar{A}q^{t_1}+\bar{B}q^{t_2}-\frac{2}{nK_0}-\frac{K\Omega^2q^2}{8(H+1)-nK}+\frac{3+\nu}{8}\Omega^2q^2, \quad (26)$$

and the nondimensional elastic-plastic interface radius $\xi$ can be found from the following equation:

$$\frac{w^2}{\left[\left(\frac{q}{\xi}\right)^{t_2}-\left(\frac{q}{\xi}\right)^{t_1}\right]}\left\{\frac{t_1-t_2}{(t_1-n)(t_2-n)}\left[\bar{I}-(1-\nu)\left[\frac{1}{K_0}+\frac{nK\Omega^2q^2}{4(8(H+1)-nK)}\right]\right]\right.$$
$$\left.+(H+1)\left[\frac{(q/\xi)^{t_1}}{t_2-n}-\frac{(q/\xi)^{t_2}}{t_1-n}\right]\left[\frac{2(1-\nu)\Omega^2\xi^2}{8(H+1)-nK}-\frac{(1+\nu)}{K_0}\right]\right\}-\frac{2}{nK_0}-\frac{K\Omega^2\xi^2}{8(H+1)-nK}$$
$$=\frac{1}{(\beta-1)\xi^{\beta_0}-(\alpha-1)\xi^{\alpha_0}}\left\{(\xi^{\beta_0}-\xi^{\alpha_0})\left[1-\frac{2(1-\nu)\Omega^2\xi^2}{8-n(3+\nu)}\right]-\frac{(3+\nu)(\alpha-\beta)\Omega^2\xi^{\alpha_0+\beta_0}}{8-n(3+\nu)}\right\}$$
$$-\frac{(3+\nu)\Omega^2\xi^2}{8-n(3+\nu)} \quad (27)$$

where $\alpha_0 = \alpha+n-1$, $\beta_0 = \beta+n-1$.

## 4. THE FULLY PLASTIC HUB

In the particular case for $\xi = 1$ the hub becomes fully plastic. $A$ and $B$ in eqns (12)–(15), and $C_1$ can be determined by the following conditions. Making use of the geometric condition $u^{hub}(a)-u^{disk}(a) = I$, the continuity of radial stress at $r = a$ and at $r = b$ $\sigma_{rr} = 0$, the unknowns are found to be:

$$\bar{A} = \frac{A}{\sigma_0 b^{-t_1}} = \frac{w^2}{(t_1-n)q^{t_1}-(t_2-n)q^{t_2}}\left\{\bar{I} - \frac{1}{K_0}\left[1-\nu+\frac{2(t_2-n)q^{t_2}}{nw^2}\right]\right.$$
$$\left. - \left[\tfrac{1}{4}(1-\nu)nq^2 + \frac{1}{w^2}(t_2-n)q^{t_2}\right]\frac{K\Omega^2}{8(H+1)-nK}\right\} \quad (28)$$

$$\bar{B} = \frac{B}{\sigma_0 b^{-t_2}} = \frac{w^2}{(t_2-n)q^{t_2}-(t_1-n)q^{t_1}}\left\{\bar{I} - \frac{1}{K_0}\left[1-\nu+\frac{2(t_1-n)q^{t_1}}{nw^2}\right]\right.$$
$$\left. - \left[\tfrac{1}{4}(1-\nu)nq^2 + \frac{1}{w^2}(t_1-n)q^{t_1}\right]\frac{K\Omega^2}{8(H+1)-nK}\right\} \quad (29)$$

$$\bar{C}_1 = \frac{C_1}{\sigma_0} = \bar{A}q^{t_1} + \bar{B}q^{t_2} - \frac{2}{nK_0} - \frac{K\Omega^2 q^2}{8(H+1)-nK} + \frac{(3+\nu)}{8}\Omega^2 q^2. \quad (30)$$

## 5. NUMERICAL RESULTS AND DISCUSSION

It can be seen from the present analysis that the elastic–plastic interface radius $z$ depends on the hardening $\eta$; the stresses and radial displacement in the outer elastic region are influenced by the occurrence of hardening. However, it is well known (Gamer, 1986; Gamer, 1987a,b,c) that the occurrence of hardening does not influence the elastic–plastic interface radius $z$ and the outer elastic region of the hub with uniform thickness.

The derivation of stress and displacement in the plastic region of the hub is based on the yield condition (1). From the work of the Gamer and Kollmann (1986) it is known that for an elastic–perfectly plastic material the stress $\sigma_{rr}$ can change its sign in the plastic region of the hub. By rotation, the level of both stresses is raised and, at a certain angular velocity, the radial stress at the elastic plastic border vanishes.

From (12), (22) and (23) there follows

$$t_0\left[\bar{I} - \frac{(1-\nu)}{K_0}\right] - \frac{(1+\nu)(H+1)}{K_0}\left[\frac{(q/\zeta)^{t_2}}{t_1-n} - \frac{(q/\zeta)^{t_1}}{t_2-n}\right] - \frac{2}{nK_0 w^2}\left[\left(\frac{q}{\zeta}\right)^{t_1} - \left(\frac{q}{\zeta}\right)^{t_2}\right]$$
$$= \left\{2(1-\nu)(H+1)\zeta^2\left[\frac{(q/\zeta)^{t_1}}{t_2-n} - \frac{(q/\zeta)^{t_2}}{t_1-n}\right] + \frac{1-\nu}{4}nt_0Kq^2\right.$$
$$\left. + K\left(\frac{\zeta}{w}\right)^2\left[\left(\frac{q}{\zeta}\right)^{t_1} - \left(\frac{q}{\zeta}\right)^{t_2}\right]\right\}\frac{\Omega^2}{8(H+1)-nK} \quad (31)$$

where

$$t_0 = \frac{t_2-t_1}{(t_1-n)(t_2-n)}.$$

It can be seen from (31) whether the condition $\sigma_{rr} < 0$ holds for all possible combinations of geometric and material data.

Numerical results are presented graphically showing the influence of hardening parameter on the distribution stress in a fully plasticized hub, for $q = 0.5$, $\Omega^2 = 1.1$ and $n = 0.5$. The Poisson ratio, $\nu$, equals 0.3. Figure 2 shows the stresses distribution in the fully plasticized hub for $H = 0.5$ and $H = 2$. Figure 3 shows the radial displacement distribution in the fully plasticized hub for $H = 0.5$ and $H = 2$.

The corresponding interferences from (27) are $\bar{I}_f = 3.1921478$ and $\bar{I}_f = 3.1614821$, respectively.

To avoid the appearance of a special plastic region at the outer edge of the fully plastic hub, the radial stress must not be allowed to become positive with increasing angular

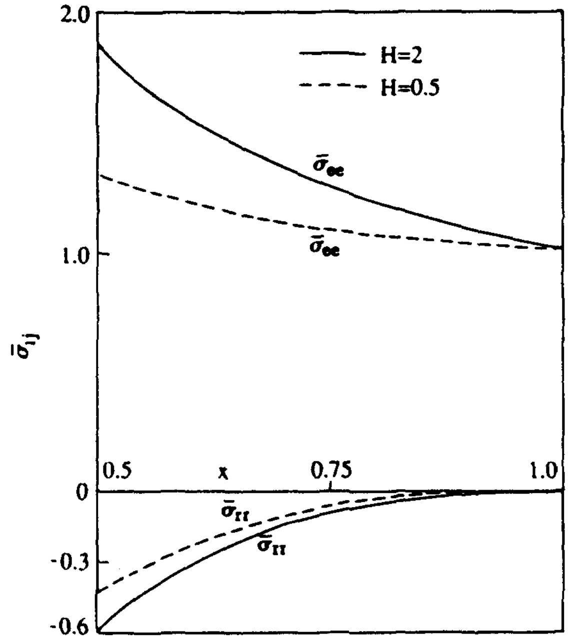


Fig. 2. Stress distribution in a fully plasticized hub.

velocity. This is guaranteed if the slope $\mathrm{d}\bar{\sigma}_{rr}/\mathrm{d}x$ at $x = 1$ is smaller than or equal to zero. From (12), (28) and (29) there follows:

$$w^2(t_1-t_2)\left[\bar{I}-\frac{(1-\nu)}{K_0}\right]+\frac{2}{nK_0}[t_2(t_1-n)q^{t_1}-t_1(t_2-n)q^{t_2}]$$
$$=\left\{(2-t_2)(t_1-n)q^{t_1}-(2-t_1)(t_2-n)q^{t_2}+\frac{1-\nu}{4}(t_1-t_2)nq^2w^2\right\}$$
$$\times\frac{K\Omega^2}{8(H+1)-nK}. \quad (32)$$

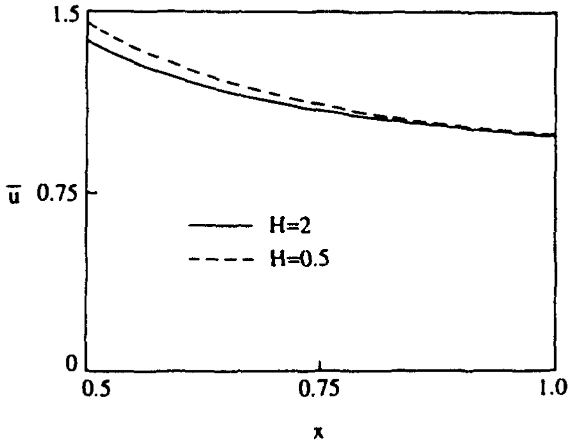


Fig. 3. Radial displacement distribution in a fully plasticized hub.

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